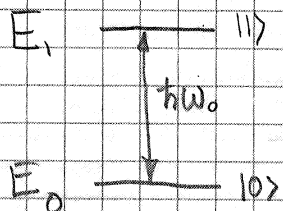


L2.1

Qubit

ex: Spin in a magnetic field

$$H = -\vec{\mu} \vec{B}$$

$$H_0 = \frac{\hbar\omega_0}{2} \sigma_z$$

Pauli Matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

state of the qubit

$$|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle \quad a_i \in \mathbb{C}$$

$$1 = \langle\psi|\psi\rangle = |a_0|^2 + |a_1|^2$$

global phase

$$|\psi\rangle = \cos(\theta/2) |0\rangle + e^{i\varphi} \sin(\theta/2) |1\rangle$$

$$x = \sin(\theta) \cos(\varphi)$$

$$y = \sin(\theta) \sin(\varphi)$$

$$z = \cos(\theta)$$

$$\text{Example: } \theta = \frac{\pi}{2} \quad \varphi = \frac{\pi}{2}$$

$$\Rightarrow (0, 1, 0)$$

$$|y\rangle = \frac{1}{\sqrt{2}} (|0\rangle + i|1\rangle)$$

Schrödinger Equation

$$i\hbar \frac{d}{dt} |\psi\rangle = H |\psi\rangle$$

time evolution of state
of a closed quantum system

$$a_0(t) = e^{-i\omega_0/2 t} a_0(0)$$

$$a_1(t) = e^{+i\omega_0/2 t} a_1(0)$$

$$\varphi(t) = \omega_0 t + \varphi(0)$$

$$\theta(t) = \theta(0)$$

Larmor precession

L2.2

External oscillating field

$$H = H_0 + \frac{W}{2} (\sigma_x \cos(\omega t) + \sigma_y \sin(\omega t))$$

example $\vec{B} = \vec{B}_0 + \vec{B}_1 \cos(\omega t)$

$$H = \frac{\hbar}{2} \begin{pmatrix} \omega_0 & \omega_1 e^{-i\omega t} \\ \omega_1 e^{i\omega t} & -\omega_0 \end{pmatrix}$$

$$\hbar \omega_1 := W$$

$$i \frac{d}{dt} a_1(t) = \frac{\omega_0}{2} a_1(t) + \frac{\omega_1}{2} e^{-i\omega t} a_0(t)$$

$$i \frac{d}{dt} a_0(t) = \frac{\omega_1}{2} e^{i\omega t} a_1(t) - \frac{\omega_0}{2} a_0(t)$$

changing to a rotating frame

$$b_1(t) = a_1(t) e^{i\omega t/2}$$

$$b_0(t) = a_0(t) e^{-i\omega t/2}$$

$$i \frac{d}{dt} b_1(t) = -\frac{\Delta\omega}{2} b_1(t) + \frac{\omega_1}{2} b_0(t)$$

$$i \frac{d}{dt} b_0(t) = \frac{\omega_1}{2} b_1(t) + \frac{\Delta\omega}{2} b_0(t)$$

$$\Delta\omega = \omega_0 - \omega$$

$$i\hbar \frac{d}{dt} |\tilde{\psi}(t)\rangle = \tilde{H} |\tilde{\psi}(t)\rangle$$

$$\tilde{H} = \begin{pmatrix} -\Delta\omega & \omega_1 \\ \omega_1 & \Delta\omega \end{pmatrix}$$

$$|\tilde{\psi}(t)\rangle = \mathcal{R}(t) |\psi(t)\rangle$$

$$\mathcal{R}(t) = e^{i\omega t \sigma_z / \hbar}$$

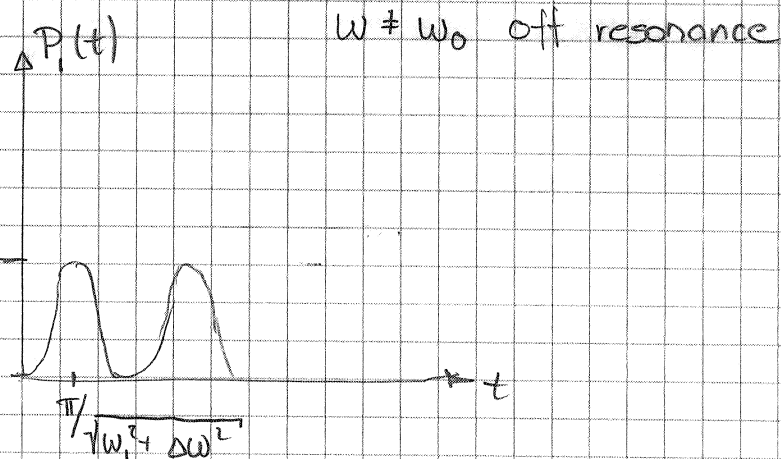
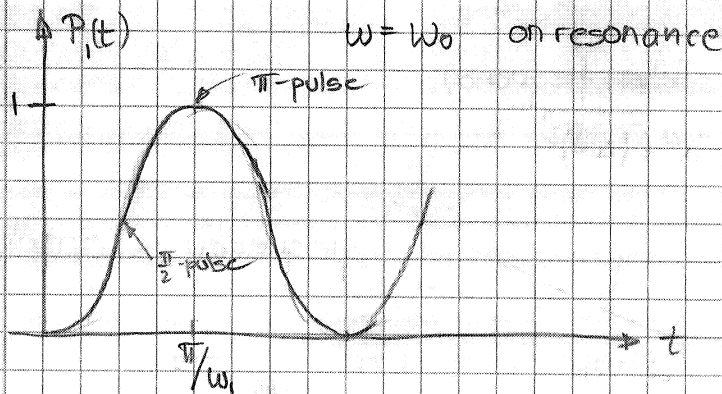
L2.3

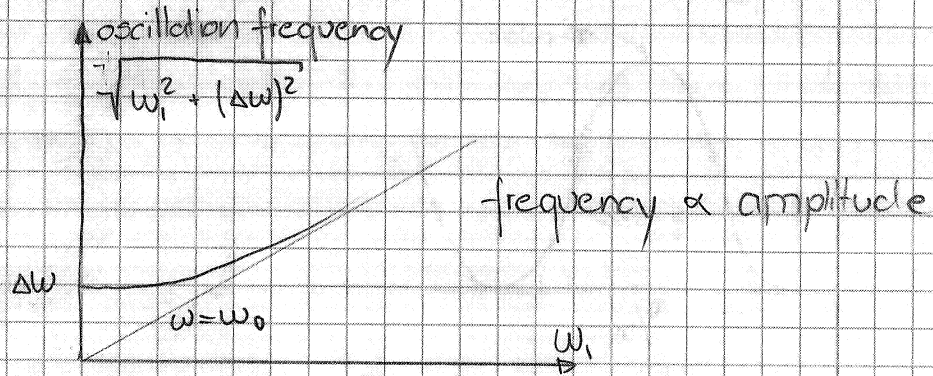
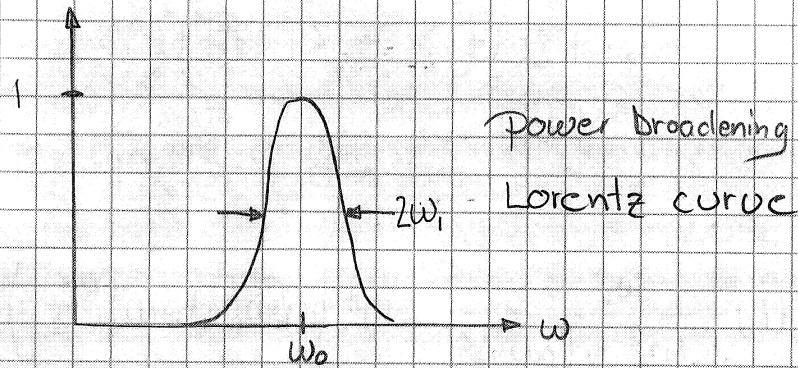
$$P_1(t) = \langle 1 | \psi(t) \rangle = \langle 1 | \tilde{\psi}(t) \rangle$$

$$= b_1(t)$$

see supplement

$$P_1(t) = \frac{\omega_1^2}{\omega_1^2 + (\Delta\omega)^2} \sin^2 \left(\sqrt{\omega_1^2 + (\Delta\omega)^2} \frac{t}{2} \right)$$





L2.4

Open quantum system

density matrix

ensemble of pure states $|\psi_i\rangle$
with probability p_i :

$$\rho = \sum p_i |\psi_i\rangle \langle \psi_i|$$

general properties

$$\boxed{\text{tr}(\rho) = 1}$$

$$\text{tr}(\rho) = \sum p_i \text{tr}(|\psi_i\rangle \langle \psi_i|) = \sum p_i = 1$$

ρ is a positive, hermitian

$$\langle \psi | \rho | \psi \rangle \geq 0 \quad \rho = \rho^\dagger$$

pure state $\rho^2 = \rho$

$$\text{tr}(\rho^2) \leq 1 \quad \text{general}$$

$$= 1 \iff \text{pure state}$$

$$< 1 \iff \text{mixed state}$$

expectation value of an operator

$$\langle A \rangle = \text{tr}(\rho A)$$

spin $\frac{1}{2}$

pure state

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} \cos^2(\theta/2) & \sin(\theta/2)\cos(\theta/2)e^{-2i\phi} \\ \sin(\theta/2)\cos(\theta/2)e^{2i\phi} & \sin^2(\theta/2) \end{pmatrix}$$

examples $|1\rangle \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$|x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \rightarrow \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\text{tr}(\rho_x^2) = 1$$

mixed state

$$\frac{1}{2} |0\rangle\langle 0| + \frac{1}{2} |1\rangle\langle 1| = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\text{tr}(S^2) = \frac{1}{2}$$

general $\rho \in \mathbb{C}(2 \times 2)$

$$\rho = \alpha_0 \mathbb{1} + \vec{\alpha} \cdot \vec{\sigma}$$

$$= \alpha_0 \mathbb{1} + \alpha_x \sigma_x + \alpha_y \sigma_y + \alpha_z \sigma_z$$

ρ density matrix

$$\rho = \frac{1}{2} (\mathbb{1} + \vec{s} \cdot \vec{\sigma})$$

$$s_x = \langle \sigma_x \rangle = \text{tr}(\rho \sigma_x)$$

$$s_y = \langle \sigma_y \rangle = \text{tr}(\rho \sigma_y)$$

$$s_z = \langle \sigma_z \rangle = \text{tr}(\rho \sigma_z)$$

$$\text{tr}(\rho^2) \leq 1 \Rightarrow s_x^2 + s_y^2 + s_z^2 \leq 1$$

= 1 pure state

dynamics

von Neumann equation

$$i\hbar \frac{dS}{dt} = [H, S] \quad [a, b] = ab - ba$$

unitary evolution

Lindblad master equation

$$\frac{dS}{dt} = -\frac{i}{\hbar} [H, S] + \sum_j \left(2L_j S L_j^\dagger - \{L_j^\dagger L_j, S\} \right)$$

markovian

memoryless

Note: $-\text{tr}(S(t)) = 1 \quad S(t) = S^\dagger(t)$

L2.6

Example: spontaneous emission

$$L = \sqrt{\gamma} \sigma_-$$

$$\rightarrow L^\dagger = \sqrt{\gamma} \sigma_+$$

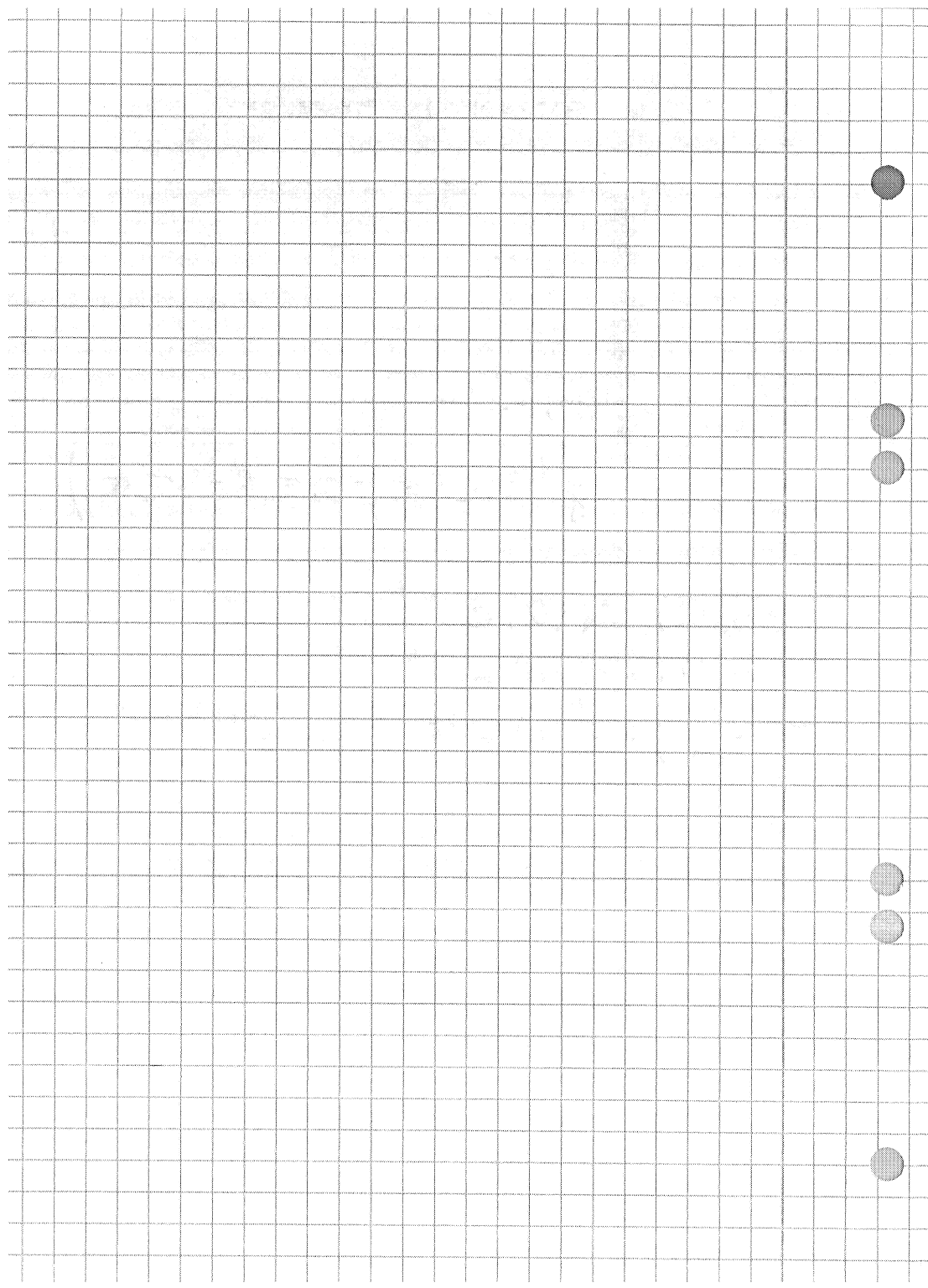
$$\frac{dS}{dt} = -\frac{i}{\hbar} [H_0, S] + \dots$$

$$+ \gamma (2S_- \rho_{11} - \sigma_+ \sigma_- \rho - \rho \sigma_+ \sigma_-)$$

$$s_x(t) = s_x(0) e^{-\gamma t}$$

$$s_y(t) = s_y(0) e^{-\gamma t}$$

$$s_z(t) = s_z(0) e^{-2\gamma t} + (1 - e^{-2\gamma t})$$



L2, S1

$$i \frac{d}{dt} |\psi\rangle = H |\psi\rangle$$

$$|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle$$

$$H = \alpha \sigma_z + \beta \sigma_x = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}$$

$$\tilde{H} = \begin{pmatrix} \sqrt{\alpha^2 + \beta^2} & 0 \\ 0 & -\sqrt{\alpha^2 + \beta^2} \end{pmatrix}$$

$$U = \begin{pmatrix} \cos(\delta/2) & \sin(\delta/2) \\ -\sin(\delta/2) & \cos(\delta/2) \end{pmatrix}$$

$$\tan(\delta) = \frac{\beta}{\alpha}$$

$$H = U^{-1} \tilde{H} U$$

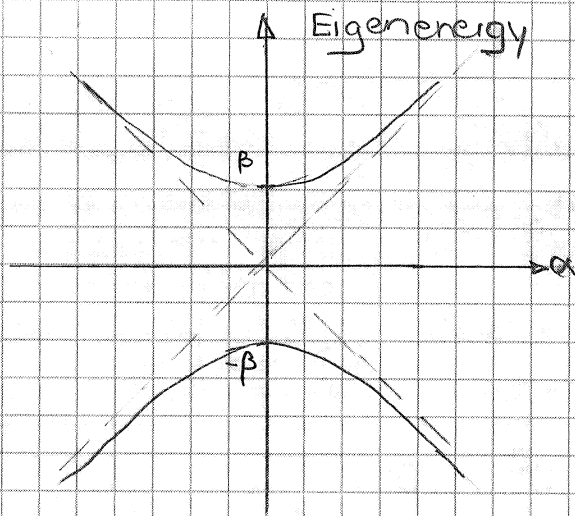
time evolution

$$|\psi(t)\rangle = \exp(-iHt/\hbar) |\psi(0)\rangle$$

let's assume that at $t=0$

the system was in state $|0\rangle$

$$|\psi(0)\rangle = |0\rangle = \cos(\delta/2) |+\rangle - \sin(\delta/2) |-\rangle$$



$$| \psi(t) \rangle = \cos(\delta/2) e^{-i\sqrt{\alpha^2 + \beta^2} t/\hbar} | + \rangle - \sin(\delta/2) e^{+i\sqrt{\alpha^2 + \beta^2} t/\hbar} | - \rangle$$

amplitude to be in the excited state

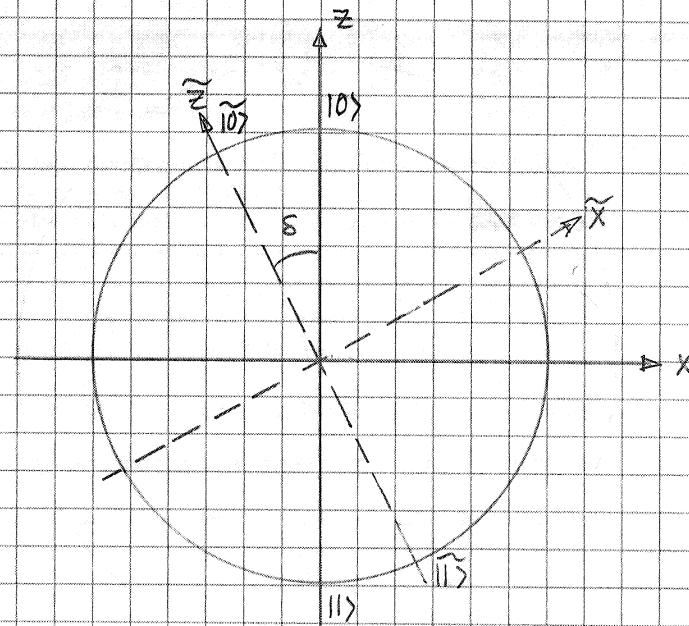
$$\langle 1 | \psi(t) \rangle = \cos(\delta/2) e^{-i\sqrt{\alpha^2 + \beta^2} t/\hbar} \langle 1 | + \rangle + \sin(\delta/2) e^{+i\sqrt{\alpha^2 + \beta^2} t/\hbar} \langle 1 | - \rangle$$

using $\langle 1 | + \rangle = \sin(\delta/2)$ and $\langle 1 | - \rangle = \cos(\delta/2)$

$$\begin{aligned} \langle 1 | \psi(t) \rangle &= \cos(\delta/2) \sin(\delta/2) \\ &\quad \cdot 2i \sin\left(\sqrt{\alpha^2 + \beta^2} t/\hbar\right) \\ &= \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} i \sin\left(\sqrt{\alpha^2 + \beta^2} t/\hbar\right) \end{aligned}$$

L2.S2

Bloch sphere visualization



Due to the off-diagonal element β the states $|0\rangle$ and $|1\rangle$ are not eigenstates anymore. The eigenstates correspond to a Bloch sphere that is rotated by the angle δ . The time evolution of the state $|0\rangle$ is a Larmor precession around the axes \tilde{z} .

